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The Projection Method for Computing Multidimensional Absolutely Continuous Invariant Measures

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We present an algorithm for numerically computing an absolutely continuous invariant measure associated with a piecewise C^2 expanding mapping $S: \Omega \to \Omega$ on a bounded region $\Omega \subset R^N$. The method is based on the Galerkin projection principle for solving an operator equation in a Banach space. With the help of the modern notion of functions of bounded variation in multidimension, we prove the convergence of the algorithm.

KEY WORDS: Frobenius–Perron operators; invariant measures.

1. INTRODUCTION

In physical science, many problems are closely related to that of the existence and computation of invariant measures for nonsingular transformations on measure spaces.[15] For one-dimensional piecewise C^2 and stretching mappings of an interval, the existence of an invariant measure which is absolutely continuous with respect to the Lebesgue measure has been proved by Lasota and Yorke.[16]

For the computation of one-dimensional absolutely continuous invariant measures, Li[17] proved the convergence of Ulam's piecewise constant approximation method for the Lasota–Yorke class of piecewise C^2 stretching mappings on $[0, 1]$. Some high-order methods have been developed.[6,7,14] A unified approach was proposed in ref. 4. Error estimates of these methods were given in refs. 2, 4, and 12. Furthermore, motivated

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by the convergence rate analysis, a systematic spectral analysis of Frobenius–Perron operators was presented in ref. 5.

For piecewise C^2 and expanding mappings in R^N, a general existence theorem was obtained in ref. 11. The theorem states that under mild conditions on the partition of the region $\Omega \subset R^N$, there exists an absolutely continuous invariant measure for $S: \Omega \to \Omega$ if the derivative of the inverse mapping $S_i^{-1}(x)$ of each piece S_i of S has norm less than some σ such that $\alpha = \sigma(1 + 1/a) < 1$ for some $a > 0$ at every point x in its domain.

Recently the convergence of Ulam's method was proved for the Jablonski class of mappings on an N-dimensional cube.[1] For general high-dimensional piecewise expanding transformations that satisfy the conditions in ref. 11, a continuous piecewise linear Markov finite approximation method was developed[8] to compute absolutely continuous invariant measures, and its convergence was established with the help of the modern notion of bounded variation. In this paper, we introduce a general high-order projection method. Compared with the Markov finite approximation method, our new algorithm employs piecewise polynomials which are not necessarily continuous to approximate the density of the invariant measure, which is in general only an L^1-function.

After giving some preliminaries in the next section, we present the first-order projection method in detail in Section 3. The outline of the second-order method is given in Section 4, and we conclude in Section 5.

2. FROBENIUS–PERRON OPERATORS AND THE PROJECTION METHOD

Let Ω be a bounded region in R^N with piecewise C^2 boundary. Throughout the paper we assume that $S: \Omega \to \Omega$ is a piecewise C^2 expanding mapping, i.e., there is a constant $0 < \sigma < 1$ and a partition $\{\Omega_1, \dots, \Omega_t\}$ of Ω such that for $i = 1, \dots, t$, Ω_i has a piecewise C^2 boundary, and the restriction $S_i = S|_{\Omega_i}$ of S on Ω_i is a C^2 mapping which can be extended to the closure of Ω_i as a C^2 mapping satisfying $\|DS_i^{-1}\| \leqslant \sigma$, where DS_i^{-1} is the derivative matrix of S_i^{-1} and $\|\cdot\|$ is the Euclidean matrix norm. S need not be continuous at a point on the boundary of Ω_i. It was shown in ref. 11 that under the above assumptions, if $\alpha \equiv \sigma(1 + 1/a) < 1$ for some constant $a > 0$, then there exists an absolutely continuous invariant measure under S.

The operator $P_S: L^1(\Omega) \to L^1(\Omega)$ defined by

$$\int_A P_S f \, dm = \int_{S^{-1}(A)} f \, dm \tag{1}$$

for every measurable subset A of Ω is called the Frobenius–Perron operator associated with S, where m is the Lebesgue measure on Ω. It is well known[15] that for $f \geqslant 0$ and $\|f\| = 1$, the absolutely continuous measure

$$\mu(A) = \int_A f\, dm \qquad \forall \text{ measurable sets } A \subset \Omega$$

is invariant under S if and only if f is a fixed point of P_S, i.e., $P_S f = f$. Here the invariance of the measure μ (under S) means that $\mu(S^{-1}(A)) = \mu(A)$ for every measurable set $A \subset \Omega$.

Some basic properties P_S are listed below without proof. For more detailed discussion of P_S, see the monograph of Lasota and Mackey [15].

Proposition 2.1. (i) P_S is a positive operator that preserves the L^1-norm of nonnegative functions. Thus P_S is a Markov operator.

(ii) $\int_\Omega P_S f\, dm = \int_\Omega f\, dm$ for $f \in L^1(\Omega)$.

(iii) Let $r > 0$ be an integer. Then $P_{S^r} = (P_S)^r$.

(iv) If $P_S f = f$, then $P_S f^+ = f^+$ and $P_S f^- = f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

Now we give a brief introduction to the Galerkin projection principle for solving an operator equation in a Banach space. Let E be a Banach space. Suppose M and N are closed subspaces of E such that $E = M \oplus N$. We can define the *projection* Q: $E \to E$ of E onto M along N as follows:

$$Qx = y \qquad \text{if} \quad x = y + z, \quad y \in M, \quad z \in N$$

Now, let E and F be two Banach spaces and let T: $E \to F$ be a bounded linear operator. We want to solve the operator equation

$$Tx = y$$

where $y \in F$ is fixed. The Galerkin projection method proceeds as follows. Choose two sequences of finite-dimensional subspaces E_n and F_n of E and F, respectively. Let Q_n be a sequence of projections from F onto F_n. In E_n we look for x_n such that

$$Q_n T x_n = Q_n y$$

Under a basis of E_n and a basis of F_n, the above equation is basically a system of algebraic equations. Thus, we can use numerical linear algebra to find approximate solutions to the original problem.

Before ending the section, we introduce the concept of functions of bounded variation in high dimensions that is essential in the convergence proof of our method. In what follows, $\|f\| \equiv \|f\|_{0,1} = \int_\Omega |f|\, dm$ is the L^1-norm of $f \in L^1(\Omega)$, and $\|g\|_{0,\infty} = \text{ess sup}\{|g(x)|: x \in \Omega\}$ is the L^∞-norm of $g \in L^\infty(\Omega)$. The following definition is given in ref. 10.

Definition 2.1. Let $\Omega \subset R^N$ be an open set and let $f \in L^1(\Omega)$. The number

$$V(f; \Omega) \equiv \int_\Omega \|Df\|\, dm = \sup \left\{ \int_\Omega f \operatorname{div} g\, dm: g \in C_0^1(\Omega; R^N), \|g\|_{0,\infty} \leqslant 1 \right\}$$

is called the variation of f over Ω, where div $g = \sum_{i=1}^N \partial g_i / \partial x_i$ and $C_0^1(\Omega; R^N)$ denotes the space of continuously differentiable mappings from Ω into R^N having compact support. If $V(f; \Omega) < \infty$, then f is said to have bounded variation in Ω. We let $BV(\Omega)$ be the Banach space of all functions in $L^1(\Omega)$ with bounded variation under the norm $\|f\|_{BV} = \|f\| + V(f; \Omega)$.

3. THE PIECEWISE LINEAR PROJECTION METHOD

In this section we look for approximate solutions of the Frobenius–Perron operator equations $P_S f = f$ in spaces of piecewise linear functions, using the projection technique. This numerical scheme is a generalization of the idea introduced in ref. 7 to the multidimensional case. For simplicity of presentation, we assume that Ω is the unit square $[0, 1]^2 \in R^2$. However, the basic idea behind our approximation method can be easily extended for general region $\Omega \subset R^N$.

Let $n > 0$ be an integer, and let $h = 1/n$ and $x_i = y_i = ih/n$ for $i = 0, 1,..., n$. Divide Ω into n^2 equal subsquares $\Omega_{ij} = I_i \times I_j = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i, j = 1,..., n$. Thus we have a uniform rectangle partition T_h of Ω.

Let Δ_h be the space of piecewise linear functions corresponding to the above partition. Then Δ_h is a $3n^2$-dimensional linear subspace of $L^1(\Omega)$. Note that $\Delta_h \subset L^\infty(\Omega)$. Let χ_A be the characteristic function of A. Then

$$\left\{ \frac{\chi_{\Omega_{ij}}}{h^2}, \frac{2(x - x_{i-1})\, \chi_{\Omega_{ij}}}{h^3}, \frac{2(y - y_{j-1})\, \chi_{\Omega_{ij}}}{h^3}: i = 1,..., n; j = 1,..., n \right\}$$

is a basis of Δ_h. Let it be ordered as $\{\varphi_k\}_{k=1}^l$ with $l = 3n^2$ in a natural way. It is obvious that each φ_k is a density, i.e., $\varphi_k \geqslant 0$ and $\|\varphi_k\| = 1$, with some Ω_{ij} as its support.

Define $Q_h: L^1(\Omega) \to \Delta_h$ by

$$\langle f - Q_h f, g \rangle = 0, \qquad \forall g \in \Delta_h$$

where $\langle f, g \rangle = \int_\Omega fg\, dm$ for $f \in L^1(\Omega)$ and $g \in L^\infty(\Omega)$. Then $\lim_{h \to 0} Q_h f = f$ for all $f \in L^1(\Omega)$ and $\|Q_h\|$ is uniformly bounded. Let $P_h = Q_h \circ P_S$; then $P_h: L^1(\Omega) \to \Delta_h$ satisfies

$$\int_\Omega P_h f\, dm = \int_\Omega f\, dm, \qquad \forall f \in L^1(\Omega)$$

Now we show that P_h has a nonzero fixed point in Δ_h.

Let the $l \times l$ matrices G and M be defined as

$$G = (g_{jk}), \qquad g_{jk} = \langle P_h \varphi_k, \varphi_j \rangle, \qquad \forall 1 \leqslant j, k \leqslant l$$

and

$$M = (m_{jk}), \qquad m_{jk} = \langle \varphi_k, \varphi_j \rangle, \qquad \forall 1 \leqslant j, k \leqslant l$$

Then, by the definition of Q_h and P_h, we have:

Lemma 3.1. P_h has a fixed point $f_h = \sum_{k=1}^{l} \xi_k \varphi_k \in \Delta_h$ if and only if $G\xi = M\xi$ for some $\xi = (\xi_1, ..., \xi_l)^T \in R^l$.

Lemma 3.2. There exists a nonzero $f_h \in \Delta_h$ such that $P_h f_h = f_h$.

Proof. Since the constant function $1 \in \Delta_h$, there exists a (nonnegative) vector $a = (a_1, ..., a_l)^T \in R^l$ such that

$$\sum_{k=1}^{l} a_k \varphi_k = 1$$

Now let $b = G^T a - M^T a$. Then the kth component of b is given by

$$\begin{aligned} b_k &= \sum_{j=1}^{l} \langle P_h \varphi_k, \varphi_j \rangle a_j - \sum_{j=1}^{l} \langle \varphi_k, \varphi_j \rangle a_j \\ &= \left\langle P_h \varphi_k, \sum_{j=1}^{l} a_j \varphi_j \right\rangle - \left\langle \varphi_k, \sum_{j=1}^{l} a_j \varphi_j \right\rangle \\ &= \langle P_h \varphi_k, 1 \rangle - \langle \varphi_k, 1 \rangle = \int_\Omega P_h \varphi_k\, dm - \int_\Omega \varphi_k\, dm = 0 \end{aligned}$$

Since the transpose of $G - M$ has a nontrivial kernel, the same is true for $G - M$. Hence, there exists a nonzero $\xi \in R^l$ such that $G\xi = M\xi$. QED

To obtain the convergence of the projection method, we need the following result (see, e.g., ref. 3).

Lemma 3.3. There exists a constant C that is independent of the partition T_h such that

$$\|Q_h f - f\| \leqslant Ch \int_{\Omega} \|\text{grad}\, f\|\, dm \qquad \forall f \in W^{1,1}(\Omega) \tag{2}$$

Now for the piecewise C^2 and expanding $S: \Omega \to \Omega$, it was proved in ref. 11 that there are two constants α and β such that for any $f \in BV(\Omega)$,

$$V(P_S f; \Omega) \leqslant \alpha V(f; \Omega) + \beta \|f\| \tag{3}$$

Lemma 3.4. There ia a number λ independent of S and h such that if $S: \Omega \to \Omega$ is piecewise C^2 and expanding such that $\lambda\alpha < 1$, then for any sequence of fixed points $\{f_h\}$ of P_h with $\|f_h\| = 1$, the sequence $\{V(f_h; \Omega)\}$ is uniformly bounded.

Proof. First of all, we show that there exists a constant λ that is independent of h such that

$$V(Q_h f; \Omega) \leqslant \lambda V(f; \Omega), \qquad \forall f \in BV(\Omega) \tag{4}$$

Let $f \in BV(\Omega)$. By Theorem 1.17 in ref. 10, there exists a sequence $\{f_j\}$ in $C^\infty(\Omega)$ such that

$$\lim_{j \to \infty} \|f_j - f\| = 0 \tag{5}$$

and

$$\lim_{j \to \infty} \int_{\Omega} \|\text{grad}\, f_j\|\, dm = V(f; \Omega) \tag{6}$$

Since $\|Q_h\|$ is uniformly bounded, (5) implies that $\lim_{j \to \infty} \|Q_h f_j - Q_h f\| = 0$. Hence, from Theorem 1.9 of ref. 10,

$$V(Q_h f; \Omega) \leqslant \liminf_{j \to \infty} V(Q_h f_j; \Omega) \tag{7}$$

Define $W_h \subset C_0^1(\Omega; R^2)$ by

$$W_h = \{g \in C_0^1(\Omega; R^2): g|_{\Omega_{ij}} \in Q_{2,3} \times Q_{3,2},\ i, j = 1,..., n\}$$

where $Q_{p,q} = \text{span}\{x^i y^j: 0 \leqslant i \leqslant p,\ 0 \leqslant j \leqslant q\}$. Then there exist $\pi_h: C_0^1(\Omega; R^2) \to W_h$ and a constant C' that is independent of the partition T_h of Ω such that (cf. ref. 9, pp. 104–111, 154–155)

$$\int_{\Omega} \text{div}(g - \pi_h g)\, f\, dm = 0, \qquad \forall f \in \Delta_h \tag{8}$$

and

$$\|\pi_h g\|_{0,\infty} \leqslant C' \|g\|_{0,\infty}, \qquad \forall g \in C_0^1(\Omega; R^2) \tag{9}$$

Now from (8), we have

$$\int_\Omega Q_h f_j \operatorname{div} g \, dm = -\int_\Omega \operatorname{grad} f_j \, \pi_h g \, dm + \int_\Omega (Q_h f_j - f_j) \operatorname{div} \pi_h g \, dm$$

Thus, from (2), (9), and the inverse estimate[3] that there is a constant C such that

$$\|\operatorname{div} \pi_h g\|_{0,\infty} \leqslant Ch^{-1} \|\pi_h g\|_{0,\infty}$$

we can find a constant λ that is independent of h and $f \in BV(\Omega)$ such that

$$\left| \int_\Omega Q_h f_j \operatorname{div} g \, dm \right| \leqslant \lambda \int_\Omega \|\operatorname{grad} f_j\| \, dm \, \|g\|_{0,\infty}$$

which implies that

$$\begin{aligned} V(Q_h f_j; \Omega) &= \sup \left\{ \int_\Omega Q_h f_j \operatorname{div} g : g \in C_0^1(\Omega; R^2), \|g\|_{0,\infty} \leqslant 1 \right\} \\ &\leqslant \lambda \int_\Omega \|\operatorname{grad} f_j\| \, dm \end{aligned} \tag{10}$$

Combining (6), (7), and (10), we obtain the conclusion.

Finally, combining (3) and (4), we have

$$V(f_h; \Omega) = V(P_h f_h; \Omega) \leqslant \lambda V(P_S f_h; \Omega) \leqslant \lambda\alpha V(f_h; \Omega) + \lambda\beta$$

Since $\lambda\alpha < 1$, we obtain

$$V(f_h; \Omega) \leqslant \frac{\lambda\beta}{1 - \lambda\alpha}$$

Thus, $\{V(f_h; \Omega)\}$ is uniformly bounded. QED

Now we can prove the convergence result of the projection method.

Theorem 3.1. Suppose $S: \Omega \to \Omega$ is piecewise C^2 and expanding such that $\lambda\alpha < 1$. Let $\{f_h\} \in \Delta_h$ be such that $P_h f_h = f_h$ and $\|f_h\| = 1$. Then there exists a subsequence $\{f_{h_k}\} \subset \{f_h\}$ such that f_{h_k} converges to a fixed point of P_S. If in addition P_S has a unique invariant density f, then we can choose f_h such that $\lim_{h \to 0} f_h = f$. Moreover, if only $\alpha < 1$, then a sequence of functions can be constructed from piecewise linear functions which converge to a fixed point of P_S.

Proof. Suppose first $\lambda\alpha < 1$. Then by Lemma 3.4, the sequence $\{f_h\}$ is bounded in $BV(\Omega)$. Theorem 1.19 of ref. 10 implies that there is a subsequence $\{f_{h_k}\} \subset \{f_h\}$ which converges to some g in $L^1(\Omega)$. Since

$$\|P_S g - g\| \leqslant \|g - f_{h_k}\| + \|f_{h_k} - Q_{h_k} \circ P_S f_{h_k}\| \\ + \|Q_{h_k} \circ P_S f_{h_k} - Q_{h_k} \circ P_S g\| + \|Q_{h_k} \circ P_S g - P_S g\|$$

noting that $Q_{h_k} \circ P_S f_{h_k} = f_{h_k}$ and $\|Q_{h_k} \circ P_S\|$ is uniformly bounded, we see that $P_S g = g$. Obviously $\|g\| = 1$.

If P_S has a unique fixed density, then (iv) of Proposition 2.1 implies that $g = f$ or $g = -f$. The above argument shows that any convergent subsequence of $\{f_h\}$ must converge to either f or $-f$. Hence, we have $\lim_{h \to 0} f_h = f$ if we put an appropriate sign to each f_h.

Now suppose $\alpha < 1$. Since in (3), α for S becomes α^r for S^r because of (iii) of Proposition 2.1, one can find an $r > 0$ such that for $\omega = S^r$ instead of S, the condition of Lemma 3.4 is satisfied. Let $f_h^{(\omega)}$ of unit length be a fixed point of $P_h(\omega)$ in Δ_h. Define

$$g_k = \frac{1}{r} \sum_{j=0}^{r-1} (P_S)^j f_{h_k}^{(\omega)}$$

where f_{h_k} is a convergent subsequence of $\{f_h\}$ from the proof of the first part of the theorem. Then g_k converges to

$$g = \frac{1}{r} \sum_{j=1}^{r-1} (P_S)^j f^{(\omega)}$$

where $f^{(\omega)}$ is a fixed point of P_ω. This g is a fixed point of P_S. In fact, from (iii) of Proposition 2.1,

$$P_S g = \frac{1}{r} \{P_S f^{(\omega)} + \cdots + (P_S)^r f^{(\omega)}\} = g \quad \text{QED}$$

4. THE PIECEWISE QUADRATIC PROJECTION METHOD

Based on the discussion in the previous section, we outline the piecewise quadratic polynomial projection method in this section. Let $\Omega \subset R^2$ be the unit square and $T_h = \{\Omega_{ij}: i, j = 1,..., n\}$ the rectangle partition of Ω with mesh h as in Section 3.

Let Ω_h be the space of piecewise quadratic functions corresponding to the above partition. Then Ω_h is a $6n^2$-dimensional linear subspace of $L^1(\Omega) \cap L^\infty(\Omega)$. A basis of Ω_h is given by

$$\left\{\frac{\chi_{\Omega_{ij}}}{h^2}, \frac{2(x-x_{i-1})\chi_{\Omega_{ij}}}{h^3}, \frac{2(y-y_{j-1})\chi_{\Omega_{ij}}}{h^3} \frac{3(x-x_{i-1})^2\chi_{\Omega_{ij}}}{h^4}, \frac{4(x-x_{i-1})(y-y_{j-1})\chi_{\Omega_{ij}}}{h^4}, \frac{3(y-y_{j-1})^2\chi_{\Omega_{ij}}}{h^4}: i=1,\ldots,n; j=1,\ldots,n\right\}$$

Let it be ordered as $\{\phi_k\}_{k=1}^{l}$ with $l=6n^2$ in a natural way.

Now let $Q_h: L^1(\Omega) \to \Omega_h$ be defined by

$$f - Q_h f, g\rangle = 0, \qquad \forall g \in \Omega_h$$

Let $P_h = Q_h \circ P_S$. Then we can show as before that there exists a number λ independent of S and h. Moreover, we have:

Lemma 4.1. There exists $f_h \in \Omega_h$ such that $P_h f_h = f_h$ and $\|f_h\| = 1$. Moreover, if $\lambda\alpha < 1$, then the sequence $\{V(f_h; \Omega)\}$ is uniformly bounded.

Now we state the convergence theorem for the second-order method.

Theorem 4.1. Suppose $S: \Omega \to \Omega$ is piecewise C^2 and expanding such that $\lambda\alpha < 1$. Let $\{f_h\} \in \Omega_h$ be such that $P_h f_h = f_h$ and $\|f_h\| = 1$. Then there exists a subsequence $\{f_{h_k}\} \subset \{f_h\}$ such that f_{h_k} converges to a fixed point of P_S. If in addition P_S has a unique invariant density f, then essentially $\lim_{h \to 0} f_h = f$. Furthermore, if only $\alpha < 1$, then a sequence of functions can be constructed from piecewise quadratic functions which converges to a fixed point of P_S.

5. CONCLUSIONS

In this paper, we presented the piecewise linear and piecewise quadratic polynomial projection methods to numerically solve the fixed-point problem of the Frobenius–Perron operator P_S associated with a high-dimensional nonsingular transformation S. For piecewise C^2 and expanding mappings S for which the existence of absolutely continuous invariant measures is guaranteed, we proved the convergence of the method, using the concept of bounded variation for functions of multi-variables.

We only described the method for the unit square in plane for the sake of simplicity of presentation. The idea of the method can be easily extended to a general region of high dimension. Moreover, the convergence result can be established in the same way.

Since in general the fixed density of P_S is only an L^1-function, the projection method using noncontinuous finite elements seems a more natural approach than the Markov approximation method with continuous finite elements used in ref. 8. On the other hand, for the Markov approximation method, the approximate fixed points f_h are guaranteed to be nonnegative, but it is not known whether this property is true for the projection method.

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